Fair Division Minimizing Inequality

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Abstract. Behavioural economists have shown that people are often averse to inequality and will make choices to avoid unequal outcomes. In this paper, we consider how to allocate indivisible goods fairly to agents with additive utilities, so as to minimize inequality. We consider how this interacts with axiomatic properties such as envy-freeness, Pareto efficiency and strategy-proofness. We also consider the computational complexity of computing allocations minimizing inequality. Unfortunately, this is computationally intractable in general so we consider several tractable mechanisms that minimize greedily the inequality. Finally, we run experiments to explore the performance of these mechanisms.

Keywords: Fair division · Gini index · Subjective Gini index · Envy index

1 Introduction

In resource allocation, one of the most frequently used normative measures of fairness is envy-freeness (i.e. no agent envies another agent's allocation). Unfortunately, when the resources are indivisible, envy-free allocations may *not* exist. In addition, computing an envy-free allocation when it exists is computationally intractable. Another desirable property in resource allocation is Pareto efficiency. In contrast to envy-free allocations, Pareto efficient allocations *always* exists. Moreover, with additive utilities, such allocations can be computed quickly. However, Pareto efficient allocations may not be very fair (e.g. giving all items to a single agent might be a Pareto efficient allocation). We consider here whether minimizing the inequality between agents offers an alternative to envy-freeness and Pareto efficiency. A number of different measures of inequality have been proposed in economics (e.g. Gini, Atkinson, Hoover indices [2,12,13]). However, we focus on the Gini index as it has been commonly used in many other settings.

We start our paper with a motivating example. We consider three normative inequality measures for fair division: the *Gini* index, the *subjective Gini* index and the *envy* index. These three indices measure the quality of allocations and

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mechanisms between perfect equality (i.e. each agent values equally their own allocation) and envy-freeness. As these are numeric measures, there are *always* allocations that minimize them. We study the relationship between the Gini, subjective Gini and envy indices and axiomatic properties such as envy-freeness, Pareto efficiency and strategy-proofness. For example, we show that there are fair division problems when *none* of the envy-free allocations minimizes the inequality indices. We further study the complexity of computing allocations minimizing each of these indices. Unfortunately, most of these computational tasks are intractable in general. For this reason, we propose three tractable online mechanisms that allocate each item in a given sequence, thus greedily minimizing the three inequality indices without the knowledge of the future items in the sequence. Finally, we run experiments with these online mechanisms.

2 Formal Background

We consider a fair division problem with agents 1 to n and indivisible items o_1 to o_m. We suppose that each agent has some (private) cardinal *utility* $u_i(o_k) \in \mathbb{R}^{\geq 0}$ for each item o_k but can submit a (public) cardinal *bid* $v_i(o_k) \in \mathbb{R}^{\geq 0}$ for each item o_k. Let A be an allocation of all items to agents. We write A_i for the bundle of items allocated to agent i, and $u_i(A_j)$ for the utility of agent i for the items in the bundle A_j. We assume additive utilities. That is, $u_i(A_j) = \sum_{o_k \in A_j} u_i(o_k)$. Additivity offers an elegant compromise between simplicity and expressivity in our model as well as in many other theoretical models of fair division (e.g. [3,7,9,14,16]). For example, in an economy, incomes and wealth are additive for the population. Also, in a food banking network, donated products accumulate additive value for the banks in the network.

We consider *responsive* mechanisms that compute actual allocations of items to agents based on their reported positive bids. We say that a mechanism is *strategy-proof* if, for each problem, no agent can increase their utility in the allocation returned by the mechanism by misreporting their bids. We are interested in welfare, fairness and efficiency properties of the allocations returned by mechanisms. The *utilitarian welfare* of A is equal to $\sum_{i \in [n]} u_i(A_i)$. The *egalitarian welfare* of A is equal to $\min_{i \in [n]} u_i(A_i)$. An allocation A is *envy-free* iff $u_i(A_i) \geq u_i(A_j)$ for every $i, j \in [n]$. An allocation A is *Pareto efficient* iff there is no allocation B such that $\forall i \in [n] : u_i(B_i) \geq u_i(A_i)$ and $\exists j \in [n] : u_j(B_j) > u_j(A_j)$.

In this paper, we study primarily how these properties are related to inequalities. One of the most frequently used measures of inequality is the *Gini* index. It is commonly used to measure inequalities in income or wealth distributions. The Gini index satisfies a number of desirable properties such as anonymity, scale independence, population independence, and the transfer principle (i.e. inequality reduces when we take from the rich and give to the poor). We will use it here to measure the inequality between agents' utilities for items in allocations. More precisely, the Gini index of an allocation equals half of the relative mean absolute difference in the utilities of the agents.

$$\text{Gini} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i(A_i) - u_j(A_j)|}{2n \sum_{i=1}^{n} u_i(A_i)}$$

The Gini index lies in the interval $[0, 1]$, taking the value 0 when all n agents get the same utility, and $1 - \frac{1}{n}$ when all agents but one agent get zero utility. In a plot of the cumulative utility distribution, the Gini index measures the ratio of the area that lies between the line of equality (i.e. all n agents get the same utility) and the Lorenz curve [10].

3 A Motivating Example

A simple example provides some motivation. Suppose Alice, Bob and Carol arrive at the car hire office and are offered to rent a Renault, a Skoda, or a Toyota car. Alice knows that Skoda's share their mechanicals with VW, and likes reliable German cars, so she prefers the Skoda most. Bob is torn between the Skoda and the more unusual Renault. And Carole loves quirky cars, so has a strong preference for the Renault. She is also an environmentalist, so dislikes VW and has a strong preference against the Skoda. Their precise utilities for the different cars are given in the following table. Who gets what car?

	Renault	Skoda	Toyota
Alice	1	8	3
Bob	8	7	1
Carol	18	1	8

There is no envy-free allocation. Bob and Carol both most prefer the Renault and only one of them can get it. The allocation with the least amount of envy (either of one person for another or in total) allocates the Renault to Carol, the Skoda to Bob and the Toyota to Alice. This is also the optimal allocation from a welfare perspective with both the greatest utilitarian and egalitarian welfare. However, Alice might not consider this allocation fair as she gets less than half the utility of Bob or Carol, as well as less than half the utility of her most preferred car, whilst Carol gets her most preferred car and Bob gets a car with value close to his greatest utility for an item.

We might decide instead that it is fairer to chose from among those allocations which minimize the inequality between Alice, Bob and Carol. For instance, allocating the Renault to Bob, the Skoda to Alice and the Toyota to Carole is one such allocation. Everyone gives their car the same 8 units of utility. This allocation is Pareto efficient and has a Gini index of 0, the minimum possible. In this allocation, only Carol envies Bob, but since she gets as much utility for her car as both Alice and Bob get for their cars, this might be acceptable.

Note that there is another allocation that minimizes inequality. Allocating the Renault to Alice, the Skoda to Carol and the Toyota to Bob gives everyone the same 1 unit of utility. This also has a Gini index of 0. However, everyone now has their least preferred car, and everyone envies everyone else. Moreover, this allocation is not Pareto efficient and has the minimal welfare possible, both from the utilitarian and egalitarian perspectives.

To sum up, this example suggests that whilst the Gini index can help in choosing between allocations, we cannot minimize inequality alone. Amongst allocations that minimize inequality, we might look to maximize welfare, minimize envy, etc. Minimizing inequality does, however, have a minor advantage over envy-freeness as a primary measure of fairness. An allocation of indivisible items minimizing inequality always exists whilst an envy-free allocation may not.

4 The Subjective Gini Index

As remarked earlier, the Gini index is typically used to measure the inequalities in income and wealth distributions. However, we are concerned here with the distribution of indivisible items that are not money, and importantly agents might have different *subjective* utilities for these items. For example, the utility you get for an item is not necessarily the same as the utility I get for it.

Should it increase the "inequality" of an allocation that someone else gets an item they value when you have little or even no value for it? To return to our motivating example, suppose Alice gets the Renault, Bob gets the Toyota, and Carol gets the Skoda. Everyone gets 1 unit of utility so this allocation has a Gini index of 0. But from everyone's subjective perspective, this is not a very equitable allocation of items. For instance, from Alice's perspective, rather than the 1 unit of utility she gets, she would get 8 units of utility for Carol's car and 3 for Bob's car. Also, from Bob's perspective, rather than the 1 unit of utility he gets, he would get 8 units of utility for Alice's car and 7 for Carol's car.

In response, we propose a *new* measure of inequality. The *subjective Gini* index takes such subjective differences into consideration. We modify the definition of the Gini index to sum the difference in utility an agent has for its allocation and the utility the *same* agent has for the allocation of items to other agents.

$$\text{subjective Gini} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i(A_i) - u_i(A_j)|}{2 \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(A_j)}$$

The subjective Gini index is between $[0, 1]$, taking the value 0 when each agent gives the same utility to each bundle of items, and $1 - \frac{1}{n}$ when one agent gets all items. Returning again to our motivating example, the allocation in which each agent gets 1 unit of utility has a Gini index of 0 but a subjective Gini index of $\frac{23}{55}$ (i.e. $\approx$0.41818181818). The allocation in which each agent gets 8 units of utility might be more preferred as it has a lower subjective Gini index of $\frac{37}{110}$ (i.e. $\approx$0.33636363636).

5 The Envy Index

Minimizing the subjective Gini index will find allocations which divide the items into bundles so that each bundle has similar utility for each agent. This reminds us of a fairness concept such as the maximin share when each agent's utility should be at least as high as the agent can guarantee by dividing the items into as many bundles as there are players and receiving their least desirable bundle [8].

On the positive side, an allocation which minimizes the subjective Gini index always exists, unlike maximin fair shares [18]. On the negative side, such an allocation may not be envy-free. To overcome this, we propose also an *envy index* whose definition is closely related to that of the subjective Gini index. Nevertheless, this index is focused only on the amount of envy in an allocation. Minimizing this index will return an envy-free allocation whenever one such exists.

$$\text{envy} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \max\{0, u_i(A_j) - u_i(A_i)\}}{\sum_{i=1}^{n} \sum_{j=1}^{n} u_i(A_j)}$$

The envy index lies in $[0, 1]$, taking the value 0 when the allocation is envy-free, and tending towards 1 as we increase the number of agents and allocate all items to just one agent. It is easy to see that the envy index is never greater (and sometimes smaller) than the subjective Gini index. Returning to our motivating example, the unique allocation minimizing the envy with index of $\frac{6}{110}$ (i.e. $\approx$0.05454545454) allocates the Renault to Carol, the Skoda to Bob and the Toyota to Alice. As we noted, this is also the optimal allocation from a welfare perspective with both the greatest utilitarian and egalitarian welfare.

6 Relationship to Envy-Freeness

We consider how these indices relate to a fairness concept such as envy-freeness. Suppose that an envy-free allocation exists. Clearly, such an allocation minimizes the envy index. On the other hand, envy-free allocations may not minimize the Gini or subjective Gini indices.

Theorem 1. *There exist problems with envy-free allocations in which* no *envy-free allocation minimizes the Gini or subjective Gini index.*

Proof. For the Gini index, let us consider the following fair division problem with 2 agents, 2 items and utilities as in the below table.

	item o_1	item o_2
agent 1	1	2
agent 2	3	1

The only envy-free allocation gives o_1 to agent 2 and o_2 to agent 1. However, the unique allocation that minimizes the Gini index gives o_1 to agent 1 and o_2 to agent 2. In this allocation, both agents envy each other. For the subjective Gini index, consider another problem with 3 agents and 3 items.

	item o_1	item o_2	item o_3
agent 1	9	1	5
agent 2	5	9	1
agent 3	1	5	9

The unique envy-free allocation gives to each agent their most valued item. However, the unique allocation that minimizes the subjective Gini index gives to each agent their second most preferred item, i.e. the one they value with utility of 5. □

The proof of Theorem 1 critically depends on the agents not sharing common (i.e. identical) utilities for items. In this case, there is no incompatibility between envy-freeness and minimizing the Gini or subjective Gini indices. If an allocation is envy-free, then every agent assigns the same utility to every bundle of allocated items.

Observation 1. *With common utilities, an allocation is envy-free iff the Gini and subjective Gini indices are zero.*

7 Relationship to Pareto Efficiency

Another fundamental notion in fair division is Pareto efficiency. We would prefer allocations where no agent can improve their outcome without making others worse off. Pareto efficiency is not necessarily compatible with minimizing inequality. The first example in the proof of Theorem 1 shows that for the Gini index. This should perhaps not be surprising as other fairness properties are also incompatible with Pareto efficiency. For example, an allocation that is envy-free may not necessarily be Pareto efficient. Moreover, each envy-free allocation could be Pareto dominated only by allocations that are not envy-free [14]. It follows quickly that minimizing the envy index is not compatible with Pareto efficiency. We can show that the same is true for the subjective Gini index.

Theorem 2. *There exist problems in which* no *Pareto efficient allocation minimizes the subjective Gini index.*

Proof. Let us consider a problem with 2 agents and 4 items. Also, let ϵ be some very small non-negative number that is strictly less than one.

	item o_1	item o_2	item o_3	item o_4
agent 1	1	$2-\epsilon$	1	ϵ
agent 2	$2-\epsilon$	1	ϵ	1

The allocation that gives o_1, o_3 to agent 1 and o_2, o_4 to agent 2 minimizes the subjective Gini index. However, the only Pareto efficient allocation swaps items o_1, o_2, giving o_1 to agent 2, and o_2 to agent 1, thus increasing the agents' utilities. □

Again, with common utilities, there is no incompatibility between Pareto efficiency and minimizing the Gini, subjective Gini and envy indices. This follows because each allocation, including those that minimize these indices, is Pareto efficient.

Observation 2. *With common utilities, any allocation that minimizes the Gini, subjective Gini or envy index is Pareto efficient.*

We can measure the trade-off between Pareto efficiency and minimizing one of these indices. The *egalitarian/utilitarian price of an index* for a given welfare is the ratio between the best welfare of any Pareto efficient allocation and the worst welfare of an allocation minimizing the index.

Theorem 3. *The utilitarian and egalitarian prices of the Gini and subjective Gini indices are unbounded.*

Proof. Consider 2 agents, 2 items and let $\epsilon \in (0, \frac{1}{2})$. Suppose the first agent gives item o_1 a utility of ϵ and o_2 a utility of $1-\epsilon$, whilst the second agent gives respectively utilities of $2-\epsilon$ and ϵ. Then, the Pareto efficient outcome with the best utilitarian and egalitarian welfare allocates o_1 to the second agent, and o_2 to the first agent. However, the only allocation that minimizes the Gini index does the reverse. The egalitarian price of the Gini index is then $\frac{1-\epsilon}{\epsilon}$ which is unbounded as ϵ goes to zero. Its utilitarian price is $\frac{3-2\epsilon}{2\epsilon}$ which is unbounded as ϵ goes to zero. The same example demonstrates that the utilitarian and egalitarian price of the subjective Gini index are also unbounded. □

For the envy index, we have examples where the utilitarian price grows as the number n of agents. We conjecture that this may also be an upper bound. And, we next show that the egalitarian price of this index is unbounded.

Theorem 4. *The egalitarian price of the envy index is unbounded.*

Proof. Let us consider the fair division problem with 3 agents and 3 items, in which the agents' utilities are as in the below table.

	item o_1	item o_2	item o_3
agent 1	1	1	1
agent 2	8	4	4
agent 3	8	4	4

The Pareto efficient outcome with the best egalitarian welfare allocates the item with utility 8 to the second or third agent, and each of the remaining items to a different agent. This has an egalitarian welfare of 1 unit. However, the allocation that minimizes the envy index gives the item with utility 8 to the second agent and both the other items to the third agent, or vice versa. As the first agent gets no items, this has an egalitarian welfare of zero units. Hence, the egalitarian price of the envy index is unbounded. □

8 Relationship to Strategy-Proofness

If we use a mechanism that minimizes one of these indices, agents have an incentive to declare false utilities. Again, this should not be too surprising. We often need to choose between fairness and strategy-proofness. For example, the random priority mechanism is strategy-proof but it can return allocations which are not envy-free [4].

Theorem 5. *A mechanism which minimizes the Gini, subjective Gini or envy index is not strategy proof.*

Proof. For the Gini index, consider the first example in the proof of Theorem 1. If agents report sincerely their utilities, the first agent gets o_1 and the second agent gets o_2. If the first agent misreports their utilities as 1/2 and 3 respectively, then the agents swap items, and both are better off. Similarly, if the second agent misreports their utilities as 2 and 1/2 respectively, then the agents swap items and are better off.

For the subjective Gini index, consider 2 agents and 4 items. Let the first agent have utilities $u_{11} = 1, u_{12} = 3/2, u_{13} = 1, u_{14} = 1/2$ whereas the second agent have utilities $u_{21} = 3/2, u_{22} = 1, u_{23} = 1/2, u_{24} = 1$. Suppose sincere play. The mechanism that minimizes the subjective Gini index to zero gives to each agent both items for which they have utility 1, or both items for which they have utilities 3/2 and 1/2. The utility of each agent is then 2. Suppose next that the first agent reports strategically bids 1, 3/2, 0, 0 for items o_1 to o_4 respectively. The mechanism now gives the first and second items to the first agent, and the third and fourth items to the second agent. The utility of the first agent is 5/2. This is a strict improvement.

For the envy index, we can use the same problem as for the subjective Gini index. □

9 Online Mechanisms

We next consider the computational properties of the Gini, subjective Gini and envy indices. Computing envy-free allocations is NP-hard even with just 2 agents, and common utilities [6]. It immediately follows that finding an allocation minimizing the envy index is NP-hard. By Observation 1, the same general result holds for the subjective Gini and Gini indices. Our approach to deal with the intractability of computing allocations that minimize inequality or envy is to use tractable online mechanisms. These will often return an allocation with little inequality or envy, even if there is no guarantee that it is minimal. These mechanisms can be applied to a given problem by picking a (perhaps random) sequence of the items. WLOG, let $o = (o_1, \ldots, o_m)$ be such a sequence. Each considered mechanism computes firstly a set of agents feasible for each next o_j in o given an allocation A of o_1 to o_{j-1}, and allocates secondly o_j to some feasible agent with a probability that is uniform with respect to the other feasible agents.

- GINI: this decides that $i \in [n]$ is feasible for o_j if $v_i(o_j) > 0$ and giving o_j to i minimizes the Gini index given A
- SUBJECTIVE GINI: this decides that $i \in [n]$ is feasible for o_j if $v_i(o_j) > 0$ and giving o_j to i minimizes the subjective Gini index given A
- ENVY: this decides that $i \in [n]$ is feasible for o_j if $v_i(o_j) > 0$ and giving o_j to i minimizes the envy index given A.

A powerful technique to study online mechanisms is competitive analysis [21]. Competitive analysis identifies the loss in efficiency due to the data arriving in an online fashion. We say that an online mechanism M is *c-competitive* for a given welfare w iff there exists a constant b such that, whatever the order o of items, we have that $w(\text{OPT}) \leq c \cdot w(M, o) + b$ holds where $w(M, o)$ is the welfare of M on o and $w(\text{OPT})$ is the optimal welfare in the offline problem. A mechanism that is c-competitive has a ratio c. Most of our mechanisms have ratios that are unbounded. For example, we can use the instance from the proof of Theorem 10 in [1] and show that both the utilitarian and egalitarian ratios of SUBJECTIVE GINI are unbounded. We next prove similar results for GINI and ENVY.

Theorem 6. *The utilitarian and egalitarian competitive ratios of* GINI *are unbounded.*

Proof. For GINI, consider the online fair division of items o_1, o_2 to agents 1,2. Further, let the utilities of the agents for the items are given in the below table in which $\epsilon \in (0, 1)$.

	item o_1	item o_2
agent 1	1	ϵ
agent 2	ϵ	1

The mechanism allocates o_1 to agent 2 and o_2 to agent 1, returning utilitarian and egalitarian welfare of 2ϵ and ϵ. The optimal offline allocation allocates o_2 to agent 2 and o_1 to agent 1, returning utilitarian and egalitarian welfare of 2 and 1. Consequently, both competitive ratios are equal to $\frac{1}{\epsilon}$ which goes to ∞ as ϵ goes to 0. □

Theorem 7. *The utilitarian competitive ratio of* ENVY *is at least $\frac{n}{2}$ whilst its egalitarian competitive ratio is unbounded.*

Proof. For the utilitarian ratio, consider n agents and n items. Let the first agent have utility n for each item, and each other agent have utility 1 for each item. Then, ENVY will allocate the first item to the first agent, and then each subsequent item to a new agent. The utilitarian welfare of this allocation is $2 \cdot n - 1$. The optimal utilitarian welfare is n^2, giving all items to agent 1.

For the egalitarian ratio, consider the online fair division of items o_1, o_2 to agents $1, 2$. Let agent 1 have a utility 1 for each item whilst agent 2 have a utility ϵ for o_1 and 0 for o_2, where $\epsilon > 0$. The mechanism allocates the items to agent 1, and thus returns an egalitarian welfare of 0. The optimal offline allocation gives to each agent an item they like, and has egalitarian welfare of ϵ. The egalitarian ratio is ∞. □

We can also measure the price of anarchy of these online mechanisms. The *price of anarchy* is closely related to the competitive ratio but now supposing agents act strategically [15]. The *price of anarchy* of an online mechanism for a given welfare is the ratio between the best welfare of an allocation returned by the mechanism when agents are sincere and the worst welfare of an allocation returned by the mechanism when agents are strategic. Interestingly, the price of anarchy of each of our online mechanisms is at least n. We conjecture that this may also be their upper bound.

Theorem 8. *The utilitarian and egalitarian prices of anarchy of* Gini, subjective Gini *and* Envy *are at least* n.

Proof. Consider an instance with n agents and n items. For $i \in \{1, \ldots, n\}$, let agent i have utility of 1 for o_i, and utility of $\epsilon > 0$ for each other item. The optimal offline allocation gives to each agent i their most valued item. The utilitarian welfare and egalitarian welfare of this allocation are n and 1 respectively.

We start with Gini. At round 1, this mechanism gives the first item to one of the agents who likes with it ϵ. The first agent then has an incentive to report at most ϵ for this item simply because they do not know what items will arrive next. By a similar argument, at round $i \in [n]$, the optimal play for agent i is to bid at most ϵ. Given this strategic profile, at the end of the allocation, each agent gets expected utility of $\frac{1}{n} + \frac{(n-1)}{n} \cdot \epsilon$. The utilitarian welfare and egalitarian welfare of this strategic allocation go to 1 and $\frac{1}{n}$ respectively as ϵ goes to zero. Consequently, the corresponding prices of this mechanism are at least n.

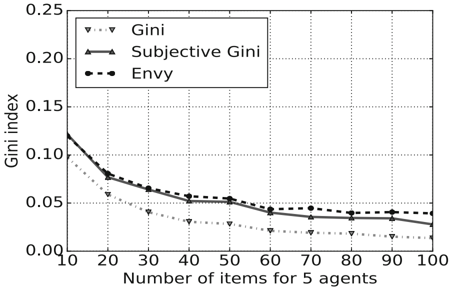
We next consider subjective Gini. The sincere play is optimal for each agent with this mechanism because they get each item with probability $\frac{1}{n}$. The welfare values go to 1 and $\frac{1}{n}$ respectively as ϵ goes to zero. The prices are at least n.

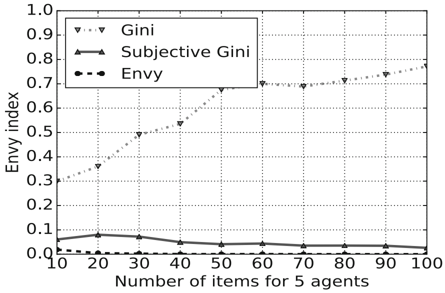
We finally consider Envy. This mechanism tends to allocate each item to agents with the highest utility for this item. By similar arguments as for Gini, we conclude that the optimal play of each agent is to bid 1 for each item. Each agent thus gets expected utility of $\frac{1}{n} + \frac{(n-1)}{n} \cdot \epsilon$. The utilitarian welfare and egalitarian welfare given this strategic profile go to 1 and $\frac{1}{n}$ as ϵ goes to zero. Hence, the prices are at least n. □

Finally, our results in this section suggest that the considered online mechanisms have performance that cannot be bounded in the worst-case. For this purpose, we next study their performance in the average-case.

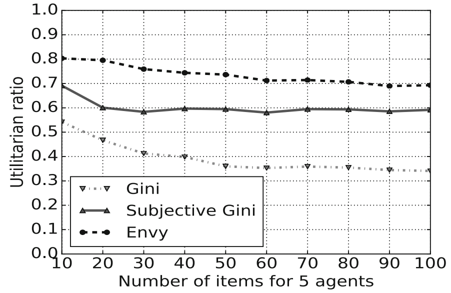
10 Experiments

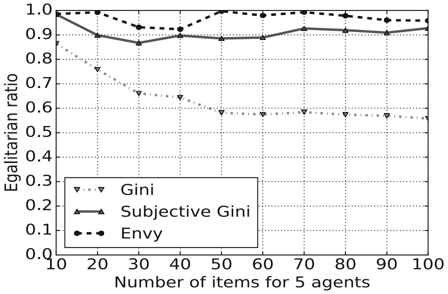
We ran a simple experiment to see how these online mechanisms would perform in practice. We generated 100 instances of $n = 5$ agents, m items for $m \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$ and integer utilities drawn uniformly at random from $\{0, 1, \ldots, m\}$. For each combination of values for n and m, we computed the Gini index, the subjective Gini index, the envy index, the egalitarian welfare and the utilitarian welfare of 100,000 sampled allocations returned by the Gini, subjective Gini and Envy mechanisms. We report in our graphs only the average results because their standard deviations were less than 1% of them. We further omit our results for the subjective Gini index for reasons of space.





In the left graph, Gini achieves the lowest value of the Gini index for each number of items. For example, the Gini value of Gini is nearly 50% lower than the Gini values of subjective Gini and Envy for 100 items. This gap actually remains similar for $m \geq 40$. Unfortunately, Gini fails to minimize envy. In the right graph, we could clearly see that Envy outperforms Gini. In fact, Envy achieves an envy index of almost 0 for 100 (and any other number m of) items. Interestingly, subjective Gini tends to favor envy-freeness to minimum inequality. By comparison, as the value of m increases, the performance of Gini diverges from envy-freeness and converges to perfect equality. Perhaps, we observe this as Gini tends to allocate items to agents with low utilities. In contrast, subjective Gini and Envy tend to allocate items to agents with high utilities, thus minimizing simultaneously both the envy and inequality.





We next discuss our results for the utilitarian and egalitarian ratios. From a utilitarian perspective (the left graph), Envy outperforms the other two mechanisms for each number of items. For example, this mechanism achieves a utilitarian ratio close to 0.7 for 100 items. This value is nearly 16% higher than the ratio of subjective Gini and 100% higher than the ratio of Gini for 100 items. From an egalitarian perspective (the right graph), again Envy outperforms subjective Gini and Gini, followed closely by subjective Gini. Interestingly, for each value of m, Envy not only minimizes the envy but also maximizes the egalitarian welfare. For 100 items, its egalitarian ratio is nearly 0.95. This value is nearly 82% higher than the value of GINI for 100 items. For both ratios, the performance of SUBJECTIVE GINI is getting closer to the performance of ENVY as the value of m increases.

Finally, our experimental results indicate that envy-freeness, (subjective) equality and welfare efficiency might often (at least approximately) be achievable in practice.

11 Related Work

Endriss has formulated the task of reducing inequality as a combinatorial optimisation problem [10]. In particular, he studied the problem of deciding if there exists an inequality reducing improvement such as a Pigou-Dalton or Lorenz transfer. The complexity of such decision problems depends on the language (e.g. the XOR-language) used to represent the (possibly non-additive) utilities. Schneckenburger, Dorn and Endriss [20] considered allocating indivisible goods to minimize inequality as measured by the Atkinson index. Their proof showing that minimizing the Atkinson index is NP-hard can be related to finding an allocation that minimizes the Gini or subjective Gini index. By comparison, we show that computing allocations with small inequalities might be fast by using online mechanisms. Other such mechanisms are used in other settings as well (e.g. [1,17]). Finally, the idea of measuring envy was firstly proposed in [11]. Moreover, there are some existing analyses of the Gini and envy indices in [5,19]. However, our idea of measuring inequality subjectively is new.

12 Conclusions

We study fair division minimizing inequality. Equitability is very important and occurs naturally in practice. For example, two people living in apartments of the same type are expected to pay equal taxes. Also, all teachers with the same qualification and experience are expected to receive the same salaries. Thus, we defined three indices that measure the quality of allocations: the Gini, subjective Gini and envy indices. The first index measures inequality within an allocation, the third one the amount of envy, whilst the second index measures the combination of both of these. We studied the relationship of these indices with envy-freeness, Pareto efficiency and strategy-proofness. Each index could be used as a second order criterion in choosing between allocations. We also proposed three tractable online mechanisms that greedily minimize these three indices. Our simple experimental results showed that, even for modest sized problems, we may be able to efficiently compute allocations with limited inequality or envy as well as with reasonably high values of the egalitarian welfare and utilitarian welfare.

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