Strategy-proofness, Envy-freeness and Pareto efficiency in Online Fair Division with Additive Utilities*

Martin Aleksandrov and Toby Walsh

Technical University Berlin, Germany
{martin.aleksandrov,toby.walsh}@tu-berlin.de

Abstract. We consider fair division problems where indivisible items arrive one by one in an online fashion and are allocated immediately to agents who have additive utilities over these items. Many existing offline mechanisms do not work in this online setting. In addition, many existing axiomatic results often do not transfer from the offline to the online setting. For this reason, we propose here three *new* online mechanisms, as well as consider the axiomatic properties of three previously proposed online mechanisms. In this paper, we use these mechanisms and characterize classes of online mechanisms that are strategy-proof, and return envy-free and Pareto efficient allocations, as well as combinations of these properties. Finally, we identify an important impossibility result.

Keywords: Online Fair Division · Strategy-Proofness · Envy-Freeness · Pareto Efficiency · Additive Utilities

1 Introduction

Fair division is an important problem facing our society today as increasing economical, environmental, and other pressures require us to try to do more with limited resources. An especially challenging form of fair division is when we are allocating available resources in an *online* fashion with only partial knowledge of the future resources and agent's preferences for these resources. There are many applications of online fair division for *social good*. For example, when a kidney is donated, it must be allocated to a patient within a few hours. As a second example, food items arrive at a food bank and must be allocated and distributed to charities promptly. As a third example, when allocating charging slots to electric cars, we may not know when or where cars will arrive for charging. As a fourth example, when managing a river, we might start allocating irrigation water to farmers today, not knowing how much it will rain the next month. As a fifth example, when allocating memory to cloud services, we may not know what and how many services are requested in the next moment.

The online nature of such fair division problems changes the mechanisms available to allocate items. For example, with the well-known (offline) *sequential allocation* mechanism, agents pick their most preferred remaining items in turns. In an online setting, an agent's most preferred item may not be currently (or even ever) available. To

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tackle this, we propose three *new* - ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY and PARETO LIKE - as well as study three existing - LIKE, BALANCED LIKE and MAXIMUM LIKE- online mechanisms. The online nature also means we may need to consider *new* axiomatic properties. For example, in deciding if agents have any incentive to misreport preferences in an online setting, we may consider the past fixed but the future unknown. This leads to a *new* and weaker form of *online strategy-proofness* (OSP). Therefore, it might be easier to achieve strategy-proofness in an online than in an offline setting. Also, we give a *new* and stronger form of envy-freeness, called *shared envy-freeness* (SEF), in which agents might be envious of each other but only over the items that they like in common. For example, in the paper assignment problem, reviewers tend to bid for papers in their field of expertise and not for papers outside this field [22]. Thus, SEF aims at guaranteeing envy-freeness across the different fields.

We provide characterization results for strategy-proofness (SP), envy-freeness (EF) and Pareto efficiency (PE). For example, we characterize completely the class of online mechanisms that are SP, and the class of online mechanisms that are PE ex post. We also characterize the class of SP and EF mechanisms. Thus, a mechanism for online fair division is SP and EF ex ante iff it returns the same random assignment as LIKE. The same holds for SEF ex ante mechanisms. Also, we prove that a mechanism is SP, PE ex post and EF ex ante iff it returns the same probability distribution of allocations as ONLINE RANDOM PRIORITY. We further give an important impossibility result. In offline fair division, stochastic Pareto efficiency and envy-freeness are always possible simultaneously (e.g. the probabilistic serial mechanism [8]). However, we prove that no online mechanism can be both Pareto efficient ex ante and envy-free ex ante.

2 Related Work

We consider the model of online fair division from [2, 3, 5, 28] in which items are indivisible and arrive one-by-one over time. We primarily contrast our characterization results with similar results in (offline) fair division. For example, we prove that *no* online mechanism can be both PE and EF ex ante. By comparison, the (offline) probabilistic serial mechanism satisfies both stochastic PE and EF [8]. In fact, it follows from our results that there could be an unbounded number of mechanisms that are just PE ex ante or EF ex ante. We can show that other (offline) characterizations (e.g. [10, 23]) break in the online setting as well. By comparison, as online mechanisms can be applied to offline problems by picking a sequence of the items, our results can be mapped into such settings. For example, our PARETO LIKE mechanism returns all possible PE ex post allocations in the offline problem. As a result, this mechanism characterizes the set of offline such mechanisms. As another example, we prove that ONLINE RANDOM PRIORITY is SP and PE ex post, but not PE ex ante. With this mechanism, agents with the same cardinal utilities receive the same expected utilities (i.e. it is symmetric). This is in-line with the impossibility result that *no* (offline or online) mechanism for offline matching is SP, PE ex ante and symmetric [30]. Yet more related results are shown in many other fair division (e.g. [6, 7, 9, 13, 18, 20, 21, 27]), voting (e.g. [14, 19, 29]) and kidney exchange (e.g. [15–17]) settings. Our results can also be mapped to such settings.

3 Online and Additive Fair Division

An online fair division *instance* consists of a set of *agents* $N = \{1, \ldots, n\}$, and an ordered set of indivisible *items* $O = \{o_1, \ldots, o_m\}$. We suppose that item o_j arrives at round j when each agent $i \in N$ becomes aware of their sincere *utility* $u_{ij} \in \mathbb{R}_{\geq 0}$ and places a possibly strategic *bid* $v_{ij} \in \mathbb{R}_{\geq 0}$ for o_j. We suppose at least one agent has positive utility for every item as, otherwise, we can simply discard the item. We use *online* mechanisms that allocate o_j immediately, supposing the allocation of o_1 to o_{j-1} is fixed and there is *no information* of o_{j+1} to o_m. We consider only *non-wasteful* mechanisms that share the probability of 1 for o_j only among agents that bid positively for it if there is at least one such agent and, otherwise, discard o_j.

An *allocation* π_j of o_1 to o_j gives a bundle of items π_{ji} to each agent $i \in N$ such that $\bigcup_{i \in N} \pi_{ji} = \{o_1, \ldots, o_j\}$ and $\pi_{ji} \cap \pi_{jk} = \emptyset$ for each $i \neq k$. We write $u_{ik}(\pi_j)$ for the *utility* of agent $i \in N$ for π_{jk}. We write $u_i(\pi_j)$ for $u_{ii}(\pi_j)$. A mechanism induces a probability distribution over the set Π_j of all allocations of items o_1 to o_j. We write $\overline{u}_{ik}(\Pi_j)$ for the *expected utility* of agent $i \in N$ for the expected allocation of agent $k \in N$ and $p_{ik}(\Pi_j)$ for the *probability* of agent $i \in N$ for item o_k in this distribution. We write $\overline{u}_i(\Pi_j)$ for $\overline{u}_{ii}(\Pi_j)$ and $p_i(\Pi_j)$ for $p_{ij}(\Pi_j)$. We suppose *additive* utilities and expected utilities.

$$u_{ik}(\pi_j) = \sum_{o_h \in \pi_{jk}} u_{ih} \qquad \overline{u}_{ik}(\Pi_j) = \sum_{h=1}^{j} p_{kh}(\Pi_j) \cdot u_{ih}$$

We consider three common properties of mechanisms: strategy-proofness, envy-freeness and Pareto efficiency.

Definition 1. *(SP) A mechanism is* strategy-proof (SP) *if, for each instance with $m \in \mathbb{N}$ items, no agent $i \in N$ can strictly increase $\overline{u}_i(\Pi_m)$ by reporting any sequence $v_{i1}, \ldots, v_{im}$ other than $u_{i1}, \ldots, u_{im}$, supposing all other agents bid sincerely for items o_1 to o_m.*

Definition 2. *(EF) A mechanism is* envy-free ex post (EFP) *iff, for each instance with $m \in \mathbb{N}$ items and allocation $\pi_m \in \Pi_m$ returned by the mechanism with positive probability, $\forall i, k \in N : u_{ii}(\pi_m) \geq u_{ik}(\pi_m)$. A mechanism is* envy-free ex ante (EFA) *iff, for each instance with $m \in \mathbb{N}$ items, $\forall i, k \in N : \overline{u}_{ii}(\Pi_m) \geq \overline{u}_{ik}(\Pi_m)$.*

Definition 3. *(PE) A mechanism is* Pareto efficient ex post (PEP) *iff, for each instance with $m \in \mathbb{N}$ items and allocation $\pi_m \in \Pi_m$ returned by the mechanism with positive probability, no $\pi'_m \in \Pi_m$ is such that $\forall i \in N : u_i(\pi'_m) \geq u_i(\pi_m)$ and $\exists k \in N : u_k(\pi'_m) > u_k(\pi_m)$. Also, it is* Pareto efficient ex ante (PEA) *iff, no mechanism gives at least $\overline{u}_i(\Pi_m)$ to each $i \in N$ and more than $\overline{u}_k(\Pi_m)$ to some $k \in N$.*

To characterize SP, EF and PE mechanisms, we will use two equivalence relations between outcomes of mechanisms. We say that two mechanisms are *ex ante equivalent* iff, for each instance of $m \in \mathbb{N}$ items, agent $i \in N$ and item $o_j \in O$, the probabilities of i for o_j under both mechanisms are equal, whilst these mechanisms are *ex post equivalent* iff, for each instance of $m \in \mathbb{N}$ items and allocation $\pi_m \in \Pi_m$, the probabilities of π_m under both mechanisms are equal (i.e. each of the two mechanisms returns an identical distribution of allocations).

4 Six Cardinal Mechanisms

Many offline mechanisms cannot be used in the online setting because only one item is available at any time. For this reason, we propose three *new* as well as study three existing online mechanisms. For every arriving item o_j, each mechanism first computes a set of agents feasible for o_j given an allocation $\pi_{j-1} \in \Pi_{j-1}$. An agent that is feasible for o_j then receives it with *conditional probability* that is uniform with respect to the other agents that are feasible for o_j. Thus, for the first j items, each mechanism returns a probability distribution over Π_j and an actual allocation with some positive probability that is obtained as a product of j conditional randomizations.

- ONLINE SERIAL DICTATOR: it has a strict priority order σ of the agents prior to round one, and the unique feasible agent for o_j is the first agent in σ that bids positively for o_j.
- ONLINE RANDOM PRIORITY: it draws uniformly at random a strict priority order σ of the agents prior to round one, and runs ONLINE SERIAL DICTATOR with it.
- PARETO LIKE: agent $i \in N$ is feasible for o_j if extending π_{j-1} by allocating o_j to i is Pareto efficient ex post.
- LIKE: agent $i \in N$ is feasible for o_j if $v_{ij} > 0$ [2].
- BALANCED LIKE: agent $i \in N$ is feasible for o_j if $v_{ij} > 0$ and i has the fewest items in π_{j-1} among those with positive bids for o_j [2].
- MAXIMUM LIKE: agent $i \in N$ is feasible for o_j if $v_{ij} = \max_{k \in N} v_{kj}$ [4].

In Example 1, we demonstrate that these mechanisms may return distributions of allocations that are different from each other.

Example 1. Let us consider an instance with $N = \{1, 2\}$ and $O = \{o_1, o_2\}$. The utilities of agents for items are given in the below table.

	item o_1	item o_2
agent 1	1	2
agent 2	2	1

In this instance, supposing sincere bidding, there are 4 possible allocations: $\pi^1 = (\{o_1, o_2\}, \emptyset)$, $\pi^2 = (\emptyset, \{o_1, o_2\})$, $\pi^3 = (\{o_1\}, \{o_2\})$, and $\pi^4 = (\{o_2\}, \{o_1\})$. ONLINE SERIAL DICTATOR with fixed $\sigma = (1, 2)$ returns π^1 with probability 1, ONLINE RANDOM PRIORITY returns π^1 and π^2 with probabilities $1/2$, PARETO LIKE returns π^1 with probability $1/2$, π^2 and π^4 with probabilities $1/4$, LIKE returns π^1 to π^4 with probabilities $1/4$, BALANCED LIKE returns π^3 and π^4 with probabilities $1/2$, and MAXIMUM LIKE returns π^4 with probability 1. □

We note that the ONLINE SERIAL DICTATOR mechanism is similar to the (offline) *serial dictatorship* mechanism [25, 26]. However, agents have no quota on the number of items they receive with ONLINE SERIAL DICTATOR, and only take items for which they declare non-zero utility. The ONLINE RANDOM PRIORITY mechanism is also similar to the (offline) *random priority* mechanism [1]. Finally, the LIKE mechanism can be seen as the online analog of the (offline) *probabilistic serial* mechanism (see [8]) with agents "eating" each next item which they like.

5 Strategy-Proofness

We begin by considering strategic behavior of agents. We provide a simple characterization of mechanisms that are strategy-proof. For $i \in N$, we say that $p_i(\Pi_j)$ is a *step* function iff it is 0 if $v_{ij} = 0$ and it admits the same value for any bid $v_{ij} > 0$ supposing the bids of the other agents for o_1 to o_j, and the bids of agent i for o_1 to o_{j-1} are fixed. A mechanism is a *step* mechanism iff, for each instance with $m \in \mathbb{N}$ items, $i \in N$ and $o_j \in O$, $p_i(\Pi_j)$ is a step function. For $i \in N$, we say that $p_i(\Pi_j)$ is a *memoryless* function iff it takes the same value for all possible bids v_{i1} to $v_{i(j-1)}$ of agent i for items o_1 to o_{j-1} given fixed bid v_{ij} of agent i for item o_j and fixed bids of the other agents for items o_1 to o_j. A mechanism is a *memoryless* mechanism iff, for each instance with $m \in \mathbb{N}$ items, $i \in N$ and $o_j \in O$, $p_i(\Pi_j)$ is a memoryless function.

With a step mechanism, $p_i(\Pi_j)$ does not depend on the size of an agent's non-zero bid for item o_j but it may depend on the allocation history. By comparison, with a memoryless mechanism, $p_i(\Pi_j)$ may depend on the size of their non-zero bid for item o_j but not on the allocation history. As a consequence, with a memoryless step mechanism, $p_i(\Pi_j)$ depends only on the combination of the non-zero bids for item o_j.

Theorem 1. *A non-wasteful mechanism for online fair division is strategy-proof iff it is a memoryless step mechanism.*

Proof. Pick $i \in N$ in an instance. Let us view $\overline{u}_i(\Pi_j)$ and $p_i(\Pi_j)$ as functions of v_{i1} to v_{ij}. That is, we write $\overline{u}_i(\Pi_j) = \overline{u}_i(v_{i1}, \dots, v_{ij})$ and $p_i(\Pi_j) = p_i(v_{i1}, \dots, v_{ij})$. Consider a memoryless step mechanism. Suppose now that all agents bid sincerely. Then, $\overline{u}_i(u_{i1}, \dots, u_{im}) = \sum_{j=1}^{m} p_i(u_{i1}, \dots, u_{ij}) \cdot u_{ij}$. Suppose next that only i bids strategically v_{i1} to v_{im}. Then, $\overline{u}_i(v_{i1}, \dots, v_{im}) = \sum_{j=1}^{m} p_i(v_{i1}, \dots, v_{ij}) \cdot u_{ij}$. For each o_j with $v_{ij} = u_{ij}$, $p_i(v_{i1}, \dots, v_{ij}) \cdot u_{ij} = p_i(u_{i1}, \dots, u_{ij}) \cdot u_{ij}$ as the mechanism is a memoryless step. For each o_j with $v_{ij} > 0$ and $u_{ij} = 0$, $p_i(v_{i1}, \dots, v_{ij}) \cdot u_{ij} = p_i(u_{i1}, \dots, u_{ij}) \cdot u_{ij} = 0$. For each o_j with $v_{ij} = 0$ and $u_{ij} > 0$, $p_i(v_{i1}, \dots, v_{ij}) \cdot u_{ij} = 0$ and $p_i(u_{i1}, \dots, u_{ij}) \cdot u_{ij} \geq 0$ as the mechanism is non-wasteful. Consequently, the mechanism is strategy-proof.

Consider a strategy-proof mechanism. First, assume that it is not a step and $p_i(u_{i1}, \dots, u_{i(j-1)}, v_{ij})$ admits different values for different positive values of v_{ij} supposing that the bids of other agents for items o_1 to o_j are fixed. WLOG, we can suppose that item o_j is the last item to arrive. We can also suppose $u_{ij} > 0$ as the case $u_{ij} = 0$ is trivial. Agent i has an incentive to report $v_{ij} > u_{ij}$ (or $v_{ij} < u_{ij}$) and, thus, strictly increase $p_i(u_{i1}, \dots, u_{i(j-1)}, u_{ij})$ and $\overline{u}_i(u_{i1}, \dots, u_{i(j-1)}, u_{ij})$. Second, assume that the mechanism is a step but not memoryless. Suppose that agent i gets different probabilities for item o_j for alternative bids v_{ik} compared to their sincere bids u_{ik} with $k < j$. WLOG, for each o_k with $k < j$, we suppose that $p_i(v_{i1}, \dots, v_{ik}) = p_i(u_{i1}, \dots, u_{ik})$. Otherwise, we truncate the problem to the first such round j. WLOG, we also suppose that $p_i(v_{i1}, \dots, v_{i(j-1)}, u_{ij}) > p_i(u_{i1}, \dots, u_{i(j-1)}, u_{ij})$. Otherwise, we swap v_{ik} for u_{ik} for $k < j$. We let agent i have utility 1 for all items except o_j and utility j for o_j. Thus, the bids v_{ik} increase the expected utility of agent i compared to the bids u_{ik}. We reached contradictions under both assumptions. □

The LIKE mechanism is a memoryless step and so is strategy-proof. We observe that the ONLINE SERIAL DICTATOR and ONLINE RANDOM PRIORITY mechanisms are also memoryless steps and, hence, are also both strategy-proof. On the other hand, the BALANCED LIKE mechanism is just a step mechanism and is neither memoryless nor strategy-proof. Furthermore, the MAXIMUM LIKE mechanism is only memoryless and the PARETO LIKE mechanism is neither a step nor a memoryless mechanism. Consequently, these two mechanisms are not strategy-proof.

Thus far, we have made the strong assumption that an agent has complete knowledge of any future items. In practice, agents may have limited or even no knowledge about the future. We next capture this formally in terms of a definition of a weaker form of strategy-proofness.

Definition 4. *(OSP) A mechanism is* online strategy-proof (OSP) *if, for each instance with $m \in \mathbb{N}$ items and $j \in \{1, \dots, m\}$, no agent $i \in N$ can strictly increase $\overline{u}_i(\Pi_j)$ by reporting any bid v_{ij} other than u_{ij}, supposing agent i bids sincerely for o_1 to o_{j-1} and all other agents bid sincerely for items o_1 to o_j.*

Indeed, it is harder for an agent to benefit from a strategic bidding with only partial information of the future. For this reason, many mechanisms that are not strategy-proof are online strategy-proof. For example, the BALANCED LIKE mechanism is online strategy-proof with no knowledge of future items, but stops being strategy-proof with complete knowledge of these future items even if all utilities are just 0 or 1 [2]. In the other direction, it is easy to show that a mechanism that is strategy-proof is also online strategy-proof. The reason for this is simple. If an agent cannot increase their expected utility by misreporting their utilities for any subset of items, then they cannot do it by misreporting their utility for any individual item, including the last one. We give a simple characterization of mechanisms that are online strategy-proof.

Theorem 2. *A non-wasteful mechanism for online fair division is online strategy-proof iff it is a step mechanism.*

Proof. We show the "if" direction. Suppose the mechanism is a step. Consider an instance, an agent $i \in N$ and an item o_j. The allocation of this item does not have an impact on the allocation of earlier items as this is now fixed. If $u_{ij} > 0$, then agent i has no incentive to report 0 for it as their expected utility can only decrease, and also has no incentive to report any positive value $v_{ij} \neq u_{ij}$ as their probability for item o_j is a step function. If $u_{ij} = 0$, then agent i has no incentive to report $v_{ij} > 0$ as their expected utility cannot increase. Hence, i cannot increase $\overline{u}_i(\Pi_j)$. The mechanism is online strategy-proof. We next sketch the "only if" direction. Suppose the mechanism is not a step. The result follows by the second part of the proof of Theorem 1. □

It follows immediately that the ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY, LIKE and BALANCED LIKE mechanisms are all online strategy-proof. In contrast, the MAXIMUM LIKE and PARETO LIKE mechanisms are not as they are not steps and agents have an incentive to report a larger bid for an item.

To sum up, we might use the ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY, or LIKE mechanism for strategy-proofness with complete information. However, for online strategy-proofness with no information about future items, we can also use the BALANCED LIKE mechanism.

6 Envy-Freeness

We continue with envy-freeness. We suppose agents bid sincerely. This might be because we use a mechanism that is strategy-proof or online strategy-proof. There is *no* envy-free ex post mechanism [2]. We, therefore, mainly focus on fairness in expectation. Uncertainty about the future means that envy-freeness ex ante is now harder to achieve than in the offline setting. Nevertheless, it is always *possible* as the LIKE mechanism is envy-free ex ante.

By Example 1, the ONLINE RANDOM PRIORITY and LIKE mechanisms can return different ex post allocations. Nevertheless, they are ex ante equivalent and, therefore, envy-free ex ante. Unfortunately, ex ante equivalence to the LIKE mechanism only provides a partial characterization as there is an unbounded number of envy-free ex ante mechanisms that are *not* ex ante equivalent to it. We show this in Example 2.

Example 2. Let us consider the fair division of items o_1 and o_2 to agents 1 and 2 with utilities as follows: $u_{11} = 1$, $u_{12} = 1$, $u_{21} = 0$ and $u_{22} = 1$. Further, consider the mechanism that works as LIKE on each instance except on this one in which it gives item o_2 to agent 2 with some probability in $(1/2, 1]$. This mechanism is envy-free ex ante but it is not ex ante equivalent to LIKE. □

In Example 2, the mechanism is neither memoryless, nor a step. Therefore, by Theorem 1, it is not strategy-proof. However, we can give a complete characterization of *all* strategy-proof and envy-free ex ante mechanisms.

Theorem 3. *A non-wasteful mechanism for online fair division is strategy-proof and envy-free ex ante iff it is ex ante equivalent to the* LIKE *mechanism.*

Proof. If a mechanism is ex ante equivalent to LIKE, then it is envy-free ex ante and a memoryless step by the definition of LIKE. By Theorem 2, the mechanism is strategy-proof. If a mechanism is envy-free ex ante and strategy-proof, then it is a memoryless step. We show that it is ex ante equivalent to LIKE by induction on the round number j. In the base case, the mechanism is clearly ex ante equivalent to LIKE. In the step case, suppose that the mechanism is ex ante equivalent to LIKE for items o_1 to o_{j-1} (i.e. hypothesis) but not for item o_j. That is, there are two agents $i, k \in N$ that like item o_j with $p_i(\Pi_j) < p_k(\Pi_j)$. As the mechanism is envy-free ex ante up to round $(j-1)$, we have that $\overline{u}_{ii}(\Pi_{j-1}) \geq \overline{u}_{ik}(\Pi_{j-1})$. As the mechanism is memoryless step, we can suppose that $u_{ij} = 1 - (\overline{u}_{ik}(\Pi_{j-1}) - \overline{u}_{ii}(\Pi_{j-1}))/(p_k(\Pi_j) - p_i(\Pi_j)) > 0$. We, hence, obtain that $\overline{u}_{ik}(\Pi_{j-1}) - \overline{u}_{ii}(\Pi_{j-1}) + (p_k(\Pi_j) - p_i(\Pi_j)) \cdot u_{ij} > 0$, or i envies ex ante k for o_1 to o_j. This contradicts the fact that the mechanism is envy-free ex ante up to round j. Consequently, $p_i(\Pi_j) = p_k(\Pi_j)$. The result follows. □

We can give similar results if we weaken strategy-proof mechanisms to memoryless or step mechanisms. We omit these proofs for reasons of space.

Proposition 1. *A step mechanism for online fair division is envy-free ex ante iff it is ex ante equivalent to the* LIKE *mechanism.*

Proposition 2. *A memoryless mechanism for online fair division is envy-free ex ante iff it is ex ante equivalent to the* LIKE *mechanism.*

On a restricted preference domain, the LIKE mechanism characterizes all envy-free ex ante mechanisms, even without the assumption of strategy-proofness. The following result applies to common domains of positive cardinal, identical cardinal, identical ordinal, Borda (e.g. $1, 2, \ldots, m$) or lexicographic (e.g. $2^0, 2^1, \ldots, 2^m$) utilities. This result holds for wasteful (i.e. not non-wasteful) mechanisms as well.

Theorem 4. *With non-zero cardinal utilities, a mechanism for online fair division is envy-free ex ante iff it is ex ante equivalent to the* LIKE *mechanism.*

Proof. We first show the "if" direction. If a mechanism is ex ante equivalent to LIKE, then it is envy-free ex ante as LIKE. We next show the "only if" direction. The proof is by induction as in Theorem 3. In the step case, we consider $i, k \in N$ that like o_j. We have that $\overline{u}_{ii}(\Pi_{j-1}) = \overline{u}_{ik}(\Pi_{j-1})$ and $\overline{u}_{kk}(\Pi_{j-1}) = \overline{u}_{ki}(\Pi_{j-1})$ as the cardinal utilities are non-zero and the mechanism is ex ante equivalent to LIKE for o_1 to o_{j-1} by the hypothesis. Hence, $p_i(\Pi_j) = p_k(\Pi_j)$ as the mechanism is envy-free ex ante up to round j. □

We can also completely characterize a stronger notion of envy-freeness even with general utilities. Shared envy-freeness requires that each pair of agents are envy-free of each other only over the items that both agents in the pair like in common. We write $u_{ik}(\pi_j)$ for the utility of agent $i \in N$ over the items in π_{ji} that both agents i and $k \in N$ like. We write $\overline{u}^{\text{SEFA}}_{ik}(\Pi_j)$ for the expected utility of agent $i \in N$ over the items o_1 to o_j that both agents i and $k \in N$ like.

$$u^{\text{SEFP}}_{ik}(\pi_j) = \sum_{\substack{o_h \in \pi_{ji} \\ u_{kh} > 0}} u_{ih} \qquad \overline{u}^{\text{SEFA}}_{ik}(\Pi_j) = \sum_{\substack{h=1 \\ u_{kh} > 0}}^{j} p_{ih}(\Pi_j) \cdot u_{ih}$$

We note $u^{\text{SEFP}}_{ik}(\pi_j) \leq u_{ii}(\pi_j)$ and $\overline{u}^{\text{SEFA}}_{ik}(\Pi_j) \leq \overline{u}_{ii}(\Pi_j)$. A mechanism is *shared envy-free ex post (SEFP)* iff, for each instance with $m \in \mathbb{N}$ items and allocation $\pi_m \in \Pi_m$ returned by the mechanism with positive probability, $\forall i, k \in N : u^{\text{SEFP}}_{ik}(\pi_m) \geq u_{ik}(\pi_m)$. A mechanism is *shared envy-free ex ante (SEFA)* iff, for each instance of $m \in \mathbb{N}$ items, $\forall i, k \in N : \overline{u}^{\text{SEFA}}_{ik}(\Pi_m) \geq \overline{u}_{ik}(\Pi_m)$. Shared envy-freeness coincides with envy-freeness with non-zero cardinal utilities. For this reason, shared envy-freeness is only possible in expectation.

Theorem 5. *A non-wasteful mechanism for online fair division is shared envy-free ex ante iff it is ex ante equivalent to the* LIKE *mechanism.*

Proof. If a mechanism is ex ante equivalent to LIKE, then it is envy-free ex ante. Every pair of agents receive each of their commonly liked item with the same probability. The mechanism is, therefore, shared envy-free ex ante. If a mechanism is shared envy-free ex ante, then the proof resembles the one of Theorem 3. In the step case, we consider round j and agents i, k that like item o_j. WLOG, assume that the mechanism is not ex ante equivalent to LIKE for item o_j and $p_i(\Pi_j) < p_k(\Pi_j)$. By the hypothesis, the mechanism is ex ante equivalent to LIKE up to round $(j-1)$. Hence, $\overline{u}_{ik}(\Pi_{j-1}) = \overline{u}^{\text{SEFA}}_{ik}(\Pi_{j-1})$ and $\overline{u}_{ki}(\Pi_{j-1}) = \overline{u}^{\text{SEFA}}_{ki}(\Pi_{j-1})$. As the mechanism is shared envy-free ex ante up to round j, $p_i(\Pi_j) = p_k(\Pi_j)$. This contradicts our assumption. □

If we limit ourselves to 0/1 utilities, we say that a mechanism is *bounded envy-free ex post with 1 (BEFP)* iff, for each instance of $m \in \mathbb{N}$ items and $\pi_m \in \Pi_m$ returned by the mechanism with positive probability, $\forall i, k \in N : u_{ii}(\pi_m) + 1 \geq u_{ik}(\pi_m)$. For example, the BALANCED LIKE mechanism is bounded envy-free ex post with 1 [2]. In fact, we can immediately conclude the following partial characterization.

Corollary 1. *With 0/1 cardinal utilities, a non-wasteful mechanism for online fair division is bounded envy-free ex post with 1 if it returns a subset of the allocations returned by the* BALANCED LIKE *mechanism.*

Benade et al. [6] showed that the random assignment of each next item (i.e. LIKE) is asymptotically optimal in the ex post sense, with a bound of the (maximum) envy that increases as the number of rounds increases. Unfortunately, this means that we cannot put any trivial bound on the envy ex post in general.

To sum up, we can use the LIKE or ONLINE RANDOM PRIORITY mechanism if we want envy-freeness ex ante. With 0/1 utilities, we can bound the ex post envy between agents to at most one unit of utility with the BALANCED LIKE mechanism which also happens to be envy-free ex ante in this domain [2].

7 Pareto Efficiency

We consider lastly Pareto efficiency supposing agents act sincerely. With 0/1 utilities, each mechanism is Pareto efficient as the sum of agents' utilities in each returned allocation is m. This is not true in general. We start with Pareto efficiency ex post. The ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY and MAXIMUM LIKE mechanisms are all Pareto efficient ex post. We might hope that a given Pareto efficient ex post mechanism returns some of the allocations returned by these three mechanisms. However, this does not hold as they may return only some of the Pareto efficient allocations. We illustrate this in Example 3.

Example 3. Let us consider the fair division of items o_1 and o_2 to agents 1 and 2 with utilities as in the below table.

	item o_1	item o_2
agent 1	1	4
agent 2	2	3

The allocation that gives o_1 to 1 and o_2 to 2 is Pareto efficient ex post. None of ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY or MAXIMUM LIKE returns this allocation. Note that PARETO LIKE does return it. □

By Example 3, we conclude that we cannot characterize all Pareto efficient ex post mechanisms in terms of allocations returned by the ONLINE SERIAL DICTATOR, ONLINE RANDOM PRIORITY and MAXIMUM LIKE mechanisms. However, we can use the PARETO LIKE mechanism for this purpose.

Theorem 6. *The* PARETO LIKE *mechanism returns only and all Pareto efficient ex post allocations.*

Proof. By definition, the mechanism returns only PE ex post allocations. For this reason, we next only show that it returns all such allocations. Consider such an allocation π_m. Assume π_m is not returned by it. Run the mechanism and follow π_m until the first round $j \in (1, m]$ when some agent $i \in N$ gets o_j in π_m but i is not feasible for o_j given the sub-allocation π_{j-1} of π_m of o_1 to o_{j-1}. Such a round exists as π_m is not returned by the mechanism. Further, π_{j-1} is Pareto efficient ex post for o_1 to o_{j-1}. Otherwise, the mechanism would not get to round j by following π_m. Also, the allocation extending π_{j-1} by allocating o_j to i is Pareto efficient ex post. Otherwise, this allocation can be Pareto improved for o_1 to o_j and together with the allocations of o_{j+1} to o_m in π_m can Pareto improve π_m. This contradicts the Pareto efficiency of π_m. Hence, the allocation extending π_{j-1} is Pareto efficient ex post. By the definition of the mechanism, it then follows that i is feasible for o_j which contradicts our assumption. Hence, π_m is returned by the mechanism with positive probability. □

By Theorem 6, we conclude that a non-wasteful mechanism for online fair division is Pareto efficient ex post iff it returns a subset of the allocations of the PARETO LIKE mechanism. Such a mechanism may not be strategy-proof. However, we can characterize *all* mechanisms that are strategy-proof and Pareto efficient ex post.

Theorem 7. *A non-wasteful mechanism for online fair division is strategy-proof and Pareto efficient ex post iff it is ex post equivalent to a probability distribution of the* ONLINE SERIAL DICTATOR *mechanisms.*

Proof. We start with the "if" direction. If a mechanism is ex post equivalent to a probability distribution of ONLINE SERIAL DICTATORS, then it is strategy-proof and Pareto efficient ex post as each ONLINE SERIAL DICTATOR. We next prove the "only if" direction. Consider a strategy-proof and Pareto efficient ex post mechanism and assume that it is not ex post equivalent to any probability distribution of ONLINE SERIAL DICTATORS. Hence, there is an instance, an allocation and $j \in [1, m]$ such that the mechanism and ONLINE SERIAL DICTATOR with some priority ordering σ agree on o_1 to o_{j-1} but the mechanism and any such ONLINE SERIAL DICTATOR disagree on o_j. WLOG, let the mechanism give o_j to 1 and ONLINE SERIAL DICTATOR with σ give o_j to 2 such that 2 is immediately before 1 in σ. Both agents like item o_j. We can show that there is o_k with $k < j$ such that 1 and 2 like o_k, and that o_k is allocated to agent 2 with both mechanisms. By Theorem 1, with the mechanism, the probabilities of 2 for o_k and 1 for o_j do not change for any positive bids of these agents for these items. WLOG, let then $u_{1j} = 1$, $u_{1k} = 2$, $u_{2j} = 2$, $u_{2k} = 1$. Hence, the allocation that extends π_{j-1} by allocating o_j to agent 1 is not Pareto efficient ex post. □

Let us next add the ex ante properties. There is an unbounded number of Pareto efficient ex post and envy-free ex ante (or Pareto efficient ex ante) mechanisms that are not strategy-proof. To see this, consider the mechanism for the instance in Example 2, that runs the ONLINE RANDOM PRIORITY (or MAXIMUM LIKE) mechanism on each other instance. Nevertheless, by Theorems 3 and 7, the only strategy-proof such mechanism is the ONLINE RANDOM PRIORITY mechanism.

Corollary 2. *A non-wasteful mechanism for online fair division is strategy-proof, Pareto efficient ex post and envy-free ex ante iff it is ex post equivalent to the* ONLINE RANDOM PRIORITY *mechanism.*

A mechanism that is Pareto efficient ex post might not be Pareto efficient ex ante. For example, the ONLINE RANDOM PRIORITY mechanism is Pareto efficient ex post but not ex ante. To see this, consider the instance in Example 1. The reverse direction may also not hold. That is, a mechanism that is Pareto efficient ex ante may not necessarily be Pareto efficient ex post. We show this in Example 4.

Example 4. Consider the mechanism that runs MAXIMUM LIKE on each instance except on the instance from Example 1. In this instance, the mechanism works as follows: agent 1 gets o_1 and o_2 with probabilities 1 and $1-\epsilon$, and agent 2 gets these items with probabilities 0 and ϵ where $\epsilon > 0$. With this mechanism, agent 1 gets expected utility $3-2\epsilon$, whilst agent 2 gets expected utility ϵ. This outcome is Pareto efficient ex ante for any $\epsilon < 1/2$. But, there is one returned allocation that gives o_1 to agent 1 and o_2 to agent 2. This outcome is not Pareto efficient ex post. □

It is easy to see that the mechanism in Example 4 is not strategy-proof. Interestingly, we can give a complete characterization of mechanisms that are strategy-proof, Pareto efficient ex post and Pareto efficient ex ante.

Theorem 8. *A non-wasteful mechanism for online fair division is strategy-proof, Pareto efficient ex post and ex ante iff it is ex post equivalent to the* ONLINE SERIAL DICTATOR *mechanism.*

Proof. We show the "if" direction. The mechanism returns the same allocation as ONLINE SERIAL DICTATOR. Hence, it is strategy-proof, Pareto efficient ex post and Pareto efficient ex ante. We next show the "only if" direction. By Theorem 7, the mechanism is a probability distribution of ONLINE SERIAL DICTATORS. Suppose that there are at least two different allocations which are the result of different ONLINE SERIAL DICTATORS in this distribution. WLOG, assume that agent 1 have the highest priority with probability $p_1 \in (0,1)$, agent 2 with $p_2 \in (0, 1-p_1]$ and agent $k \in N \setminus \{1,2\}$ with $p_k \in [0, 1-p_1-p_2]$. Suppose that agent $i \in \{1,2\}$ likes all items with 1 except o_i which they like with u, and agent $k \in N \setminus \{1,2\}$ likes items positively. The expected utility of agent $i \in \{1,2\}$ is $p_i \cdot (n-1+u)$ and the one of agent $k \in N \setminus \{1,2\}$ is p_k multiplied by the sum of their utilities. Consider now another distribution of allocations, in which agent $i \in \{1,2\}$ gets p_i for each item they like with 1 except items o_1, o_2, p_1+p_2 for item o_i and 0 for $o \in \{o_1,o_2\} \setminus \{o_i\}$ whereas agent $k \in N \setminus \{1,2\}$ gets p_k for each item. This allocation Pareto improves the allocation of the mechanism for $u > \max\{(p_1/p_2),(p_2/p_1)\}$. Hence, the mechanism is not Pareto efficient ex ante. Therefore, p_1 and p_2 cannot be both positive and, for this reason, each mechanism in the distribution gives the highest priority to the same agent. We can inductively show this for each priority. □

We next observe one last difference to the offline setting where stochastic Pareto efficiency and envy-freeness are always possible [8]. In online fair division, *no* mechanism (even wasteful) satisfies Pareto efficiency ex ante and envy-freeness ex ante unless we consider simple 0/1 utilities (e.g. the BALANCED LIKE mechanism).

Theorem 9. *With general cardinal utilities,* no *mechanism for online fair division is envy-free ex ante and Pareto efficient ex ante.*

Proof. Consider an envy-free ex ante mechanism and the instance with non-zero utilities in Example 1. By Theorem 4, to ensure envy-freeness ex ante for o_1, the mechanism should give it to each agent with $1/2$. By Theorem 4, to ensure envy-freeness for both o_1 and o_2, the mechanism then should give o_2 to each agent with $1/2$. The expected utility of each agent is $3/2$. This expected allocation is Pareto dominated by the allocation in which each agent gets the item they value with 2. Hence, the mechanism is not Pareto efficient ex ante. □

To sum up, we might use the ONLINE RANDOM PRIORITY or PARETO LIKE mechanism for Pareto efficiency ex post, or the MAXIMUM LIKE or ONLINE SERIAL DICTATOR mechanism for Pareto efficiency ex ante. With 0/1 utilities, we may also use the LIKE or BALANCED LIKE mechanism.

8 Conclusions

We summarize all results in Table 1 and Figure 1. For completeness, we add some simple results for the case of identical utilities when the PARETO LIKE and MAXIMUM LIKE mechanisms become ex post equivalent to the LIKE mechanism, the BALANCED LIKE mechanism becomes ex ante equivalent to the LIKE mechanism, and each of these becomes Pareto efficient as the sum of agents' utilities is a constant in each allocation.

Table 1. Axiomatic results. Key: ⋆ - the result follows from [Aleksandrov *et al.*, 2015].

mechanism	SP	OSP	EFA	SEFA	EFP	SEFP	BEFP	PEA	PEP
	general cardinal utilities								
ONLINE RP	✓	✓	✓	✓	×	×	×	×	✓
ONLINE SD	✓	✓	×	×	×	×	×	✓	✓
MAXIMUM LIKE	×	×	×	×	×	×	×	✓	✓
PARETO LIKE	×	×	×	×	×	×	×	×	✓
LIKE	✓⋆	✓	✓⋆	✓	×⋆	×	×⋆	×	×
BALANCED LIKE	×⋆	✓	×⋆	×	×⋆	×	×⋆	×	×
	identical cardinal utilities								
LIKE	✓⋆	✓	✓⋆	✓	×⋆	×	×	✓	✓
BALANCED LIKE	×	✓	✓	✓	×⋆	×	×	✓	✓
	binary cardinal utilities								
LIKE	✓⋆	✓	✓⋆	✓	×⋆	×	×⋆	✓	✓
BALANCED LIKE	×⋆	✓	✓⋆	×	×⋆	×	✓⋆	✓	✓

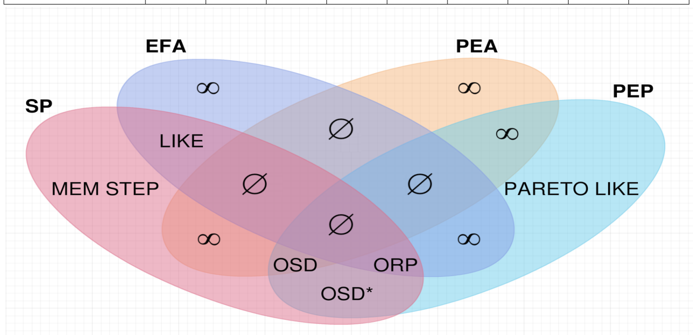

Fig. 1. General characterization results. Key: ∅ - no mechanisms, ∞ - inf. many mechanisms.

In future work, we will add quotas to our setting as in some offline settings (e.g. [20]). And, we will extend our results to approximations of envy-freeness (e.g [11, 12]) and general monotone utilities (e.g. [24]).

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