Complexity & Symmetry

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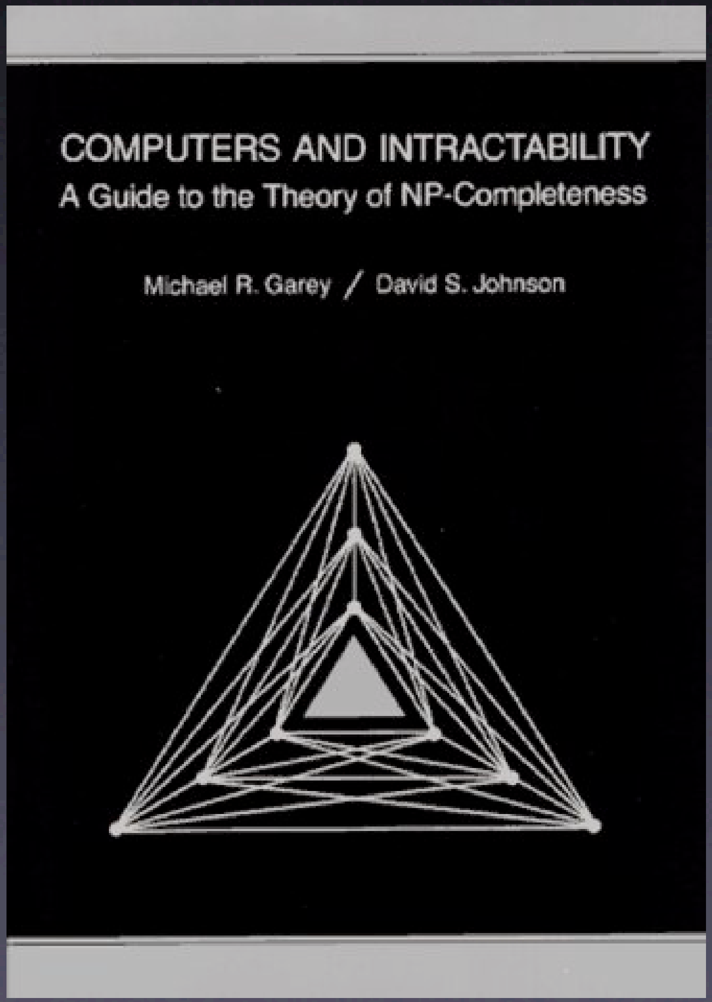
Global constraints

- Capture common patterns
  - Alldifferent(X1,…,X*n*)
  - Nvalues(N,X1,..,X*n*)
  - Lex([X1,..,X*n*],[Y1,..,Y*n*])

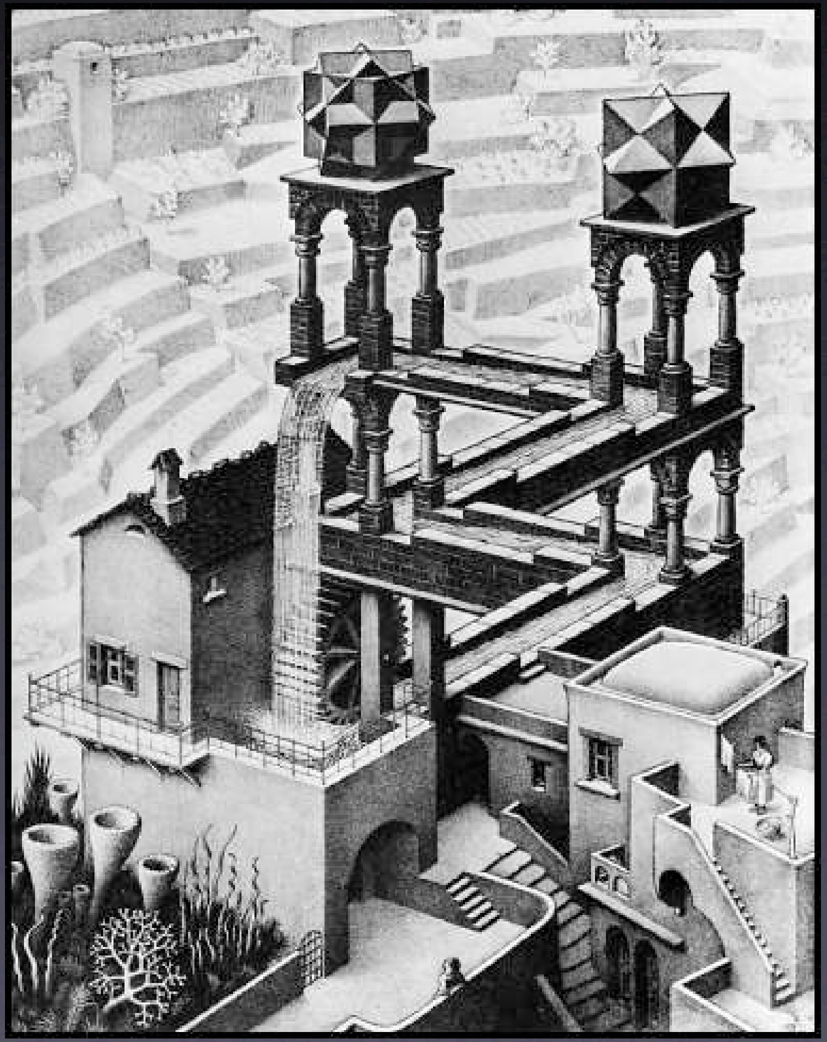
- Efficient & effective specialized propagators
  - Prune parts of search tree

Our hammer

- Use basic tools of computational complexity to study limits of
  - reasoning with global constraints
  - global constraints for breaking symmetry



Limits of Global Constraints



- Enforce lesser consistency
- Constraints cannot be combined
- Constraints cannot be generalized
- Decomposition will hurt pruning

In general

- Consider (generalized) arc-consistency
  - Every value for every variable can be extended to satisfy the constraint
  - That is, every value has *support*

- Similar results for other local consistencies
  - Bounds consistency for integer variables
  - Bounds consistency for set variables

  …

In general

- Global constraints are intractable
  - ACSupport? is NP-complete
    - Does this value have support?
- Consider C(X1,..,Xn)
  - Where Xi=j implies Xj=true, Xi=-j implies Xj=false
  - SAT in k vars, j clauses → C(X1,..,Xj+k)
    - X1 to Xk ∈ {true,false}
    - ith clause is x1 v -x3 v x5 → Xk+i ∈ {1,-3,5}
    - Consider reduction of: {x1, -x1 v x2}

In general

- Global constraints are intractable
  - ACSupport? is NP-complete
    - Does this value have support?
  - MaxAC? is DP-complete
    - Are these domains the maximal arc-consistent domains?
    - DP is NP ∪ coNP
      - Answers NP question: are these domains AC? Yes!
      - Answers coNP questions: is any smaller domain AC? No!

In general

- Global constraints are intractable
  - ACSupport? is NP-complete
  - MaxAC? is DP-complete
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- Even some specific constraints proposed in the past are intractable
  - NValues(N,X1,..X*n*)
  - AtMost1(S1,..,S*n*)
  - …

NValues

- NValues(N,X1,..,X*n*)
  - N values used in X1,…,X*n*
  - Useful for resource allocation
- Simple reduction of SAT to NValues
  - SAT problem in *k* vars, *j* clauses
  - X*i* = {*i*,-*i*} for $1 \leq i \leq k$
  - X*k*+*i* = {1,-3,5} if *i*-th clause is: (x1 v -x3 v x5)
  - N = {*n*}
  - Consider reduction of: {x1, -x1 v x2}

NValues

- NValues(N,X1,..,Xn)
  - N values used in X1,…,Xn
  - Useful for resource allocation
- Simple reduction of SAT to NValues
  - Finding support (and hence enforcing arc-consistency) is NP-hard
  - Look to enforce ***lesser*** level of local consistency like bound consistency

Composing constraints

- Take two tractable constraints
  - E.g. Disjoint(S1,…,S*n*) and FixedCard(S1,…,S*n*)
- Could we combine them into one *bigger* global constraint?
  - E.g. FixedCardDisjoint(S1,…,S*n*)
  - No, NP-hard to propagate!

GCC

- Take a tractable constraint
  - E.g. GCC([X1,..,X*n*],[l1,..,l*m*],[u1,..,u*m*])
  - Value j occurs between lj and oj times in X1,..,X*n*
- Generalize some constants to variables
  - E.g. GCC([X1,..,X*n*],[O1,..,O*m*])
  - Value j occurs Oj times in X1,..,X*n*
- NP-hard to make generalized arc-consistent
  - [Claude-Guy Quimper 2003]

GCC

- Reduction of 1in3 SAT on +ve clauses to GCC
  - If ith clause is x1 v x3 v x5 then Xi∈{1,3,5}
  - Oj∈{0,k} where k is number of occurrences of xj in clauses
  - Consider {x1 v x2 v x4, x2 v x3 v x4, x1 v x3 v x4}

Decomposing constraints

- Consider a global constraint that is NP-hard to propagate
  - E.g. AtMost1(S1,…,Sn)
- Consider a decomposition into smaller constraints
  - $|Si \cap Sj| \leq 1$ for all $i<j$
- If it is polynomial to propagate decomposition
  - decomposition must hinder propagation (assuming $P \neq NP$)

Symmetry breaking

- Add (global) constraints to eliminate symmetries
  - E.g. lex order rows, lex order cols
- Can we break *all* row & col symmetry with a single global constraint?
  - Enforcing GAC on such a global constraint is NP-hard

Conclusions

- Computational complexity is a useful hammer to study global constraints
- Uncovers fundamental limits of reasoning with global constraints
  - Lesser consistency needs to be enforced
  - Decomposition hurts pruning
  - Composition or generalization intractable
  - Symmetry breaking is inherently limited

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