Symmetry Breaking with Set Variables

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Variables

- Finite domain variables
  - Italy $\in$ {red,blue,green}
  - Square[1,2] $\in$ {white, black, empty}
- Set variables
  - Group[2,3] $\subseteq$ {player1, player2, .., player8}
  - |Group[2,3]| = 4
  - Group[2,3] $\cap$ Group[3,3] = {}

Set variables

- Explicit domain
  - But exponential number of subsets!
- Upper and lower bound
  - {} ⊆ Group[2,3] ⊆ {player1, player2, .., player8}
  - Cannot represent disjunctive choice like S={1,2} or S={2,3}

Set variables

- Characteristic function
  - X[i] = 1 iff i $\in$ S and 0 otherwise
  - Essentially equivalent to bound representation

Set variables

- More exotic representations
  - Cardinality and bounds
  - Length lex bounds
    - First order by cardinality and then, within each length, a lex ordering
    - $\{1,2,3\} \leq_{lex} S \leq_{lex} \{1,2,6\}$

Local consistency

- Bound consistency
  - Upper bound are values in a solution
  - Lower bound are values occur in all solutions
  - Equivalent to BC (GAC) on characteristic function representation

Why use set vars?

- Eliminate symmetry in a problem
  - Set has no order!
  - X[i,j,k]=1 iff golfer k plays in group i on week j
  - S[i,j] = set of golfers playing in group i on week j

Why use set vars?

- Eliminates (some) symmetry in a problem
  - Still may have symmetry between set variables
    - S[i,j] has symmetry as rows (groups) and cols (weeks) are symmetric
  - Still may have symmetry between values
    - Players (values) are interchangeable

Symmetry and Set Vars

- Finite domain variables
  - Symmetry is bijection, σ on assignments
  - σ(X[1]=4) ⇒ X[8]=3
- Set variables
  - Symmetry is bijection, σ on membership constraints
  - σ(player1 ∈ Group[3,1]) ⇒ player3 ∈ Group[1,5]
  - σ(nurse1∈ Shift[mon]) ⇒ nurse3 ∈ Shift[tu]

Symmetry and Set Vars

- Set variables
  - Symmetry is bijection, σ on membership constraints
  - Preserves solutions/constraints
- Symmetry can act on
  - Set variables alone
  - Values taken by set variables alone
  - Or both

Set variable symmetry

- Wreath value interchangeability
  - Group[i,j] is ith group in jth week
    - Weeks interchangeable
    - Given week, groups interchangeable

Set value symmetry

- Value interchangeability
  - Group[i,j] is ith group in jth week
    - Players within group interchangeable
    - Uniformly swap player3 with player4

Symmetry breaking

- General method for variable symmetries on finite domain vars [Crawford, Ginsberg, Luks and Roy KR96]
  - Look for lexicographically least assignment
    - $(Z[1],Z[2],...) \leq_{lex} (Z[\sigma(1)],Z[\sigma(2)],...)$
    - reversal symmetry:
      $(X[1],X[2],...,X[n-1],X[n]) \leq_{lex} (X[n],X[n-1],...,X[2],X[1])$

Adding constraints

- Same method works with value and variable/ value symmetries for finite domain vars [Walsh CP06]
  - Look for lex least assignment
  - For value symmetries: $(Z[1],Z[2],...) \leq_{lex} (\sigma(Z[1]),\sigma(Z[1]),...)$
  - Simple propagator for this global constraint based on a ternary decomposition

Symmetry breaking

- Same method works for set variables
  - Look for lexicographically least assignment
    - $(S[1],S[2],...) \leq_{lex} (S[\sigma(1)],S[\sigma(2)],...)$
    - But how do we order two set variables?
      - So we can lift this to lex ordering on sequence of set vars

Ordering sets

- Need any total order on sets
  - Subset is only a partial ordering
- Multiset ordering
  - $\{1,2,3\} <_{mset} \{1,2,4\}$
  - $\{1,2,3\} <_{mset} \{4\}$

Multiset ordering

- S1 $<_{mset}$ S2 iff
  - S1 can be obtained from S2 by replacing one or more values with any number of smaller values
  - Equivalent to lex ordering characteristic functions of sets
    - Suggests how to build a propagator!

Multiset ordering

- M1 $<_{mset}$ M2 iff
  - M1 can be obtained from M2 by replacing one or more values with any number of occurrences of smaller values
  - Equivalent to lex ordering occurrence vectors for multi-sets
  - $\{1,1,1,2,4,4,4,4\} <_{mset} \{4,4,5\}$

Multiset ordering constraint

- To propagate constraint S1 $<_{mset}$ S2
  - Channel into characteristic function
    - $i \in S$ iff $Xi=1$ (and 0 otherwise)
  - Post lex ordering constraint on 0/1 vars making up characteristic function
    - Consider $\{2\} \subseteq S1 \subseteq \{2,4\}$, $\{1\} \subseteq S2 \subseteq \{1,3\}$, S1 $<_{mset}$ S2

Lifting multiset ordering constraint

- To break symmetry, post LEX LEADER:
  - $(S[1],S[2],...) \leq_{lex} (S[\sigma(1)],S[\sigma(2)],...)$
  - Where $\leq_{lex}$ is lifting of multiset ordering on sets to ordering on sequences of sets
  - How do we do such a lifting?
    - Adapt $\leq_{lex}$ propagator
    - Simple encoding based on definition of $\leq_{lex}$

Lifting multiset ordering constraint

- Suppose (S[1],S[2],...) $\leq_{lex}$ (T[1],T[2],...)
  - Where S[i] and T[j] are set vars
- Introduce sequence of Booleans
  - B[i]=0 if not lex ordered up to the ith element of the sequence
  - B[i+1] iff (B[i] or S[i] $<_{mset}$ T[i])
  - B[i]=0 implies S[i] $\leq_{mset}$ T[i]

Breaking symmetry with set vars

- Look for lexicographically least assignment
  - $(S[1],S[2],...) \leq_{lex} (S[\sigma(1)],S[\sigma(2)],...)$
- Consider reversal symmetry
  - $(S[1],S[2],...) \leq_{lex} (S[n],S[n-1],...)$
- As before, may be exponential number of such constraints
  - Look for special classes of symmetry where we can do better

Interchangeable set vars

- LEX LEADER constraints imply multiset ordering on set variables
  - $S[1] \leq_{mset} S[2] \leq_{mset} \ldots \leq_{mset} S[n]$
  - Simple way to break all symmetry!
  - Consider $\{2\} \subseteq S[1] \subseteq \{2,4\}, \{1\} \subseteq S[2] \subseteq \{1,3\}, \{\} \subseteq S[3] \subseteq \{1,4\}$

Interchangeable set vars

- Symmetry breaking equivalent to row symmetry on 2d 0/1 matrix
  - Lex order rows
  - Lex chain prunes all symmetric values
  - Consider again
    - {2}⊆S[1]⊆{2,4}, {1}⊆S[2]⊆{1,3}, {}⊆S[3]⊆{1,4}

Interchangeable set vals

- Symmetry breaking equivalent to col symmetry on 2d 0/1 matrix
  - Lex order cols (nb no row sum=1 as with finite domain vars with val sym!)
  - Lex chain prunes all symmetric values
  - Consider $\{1\} \subseteq S[1] \subseteq \{1,2,3\}$, $S[2]=\{2\}$
  - Equivalent to value precedence [Law & Lee CP04]

Value precedence for set variables

- How do we distinguish apart values?
  - One value occurs in a set on its own with the other value
- Value precedence ensures that:
  - i occurs on its own first before j for all $i<j$ [Law & Lee CP04]
  - Consider $\{1\} \subseteq S[1] \subseteq \{1,2,3\}$, $S[2]=\{2\}$

Interchangeable set variables & values

- Symmetry breaking equivalent to row & col symmetry on 2d 0/1 matrix
  - NP-hard to break all symmetry
  - Lex order row & cols breaks most symmetry
  - Effective in practice

Conclusions

- Set variables help deal with symmetry
  - No order within a set
- We can break symmetry in problems containing set variables
  - In much the same way as finite domain variables
  - Look for LEX LEADER
- Special types of symmetry can do better

Questions?